ANISOTROPIC CONTINUOUS MEDIA, IN WHICH ENERGY AND THE STRESSES DEPEND ON THE GRADIENTS OF THE STRAIN TENSOR AND OTHER TENSOR QUANTITIES

(ANIZOTROPNYE SPLOSHNYE SREDY, ENERGIIA I NAPRIAZHENIIA V KOTORYKH ZAVISIAT OT GRADIENTOV TENZORA DEFORMATSII I DRUGIKH TENZORNYKH VELICHIN)

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Anistorpic continuous media in which energy and the stresses depend on the gradients of the strain tensor and other tensor quantities, are investigated.

In par. 1 we study the relationship between the derivatives of tensor characteristics with respect to coordinates of the initial, undeformed state and the derivatives with respect to coordinates of the deformed state.

In par. 2 we formulate the basic assumptions under which a closed system of equations is obtained for the unknown functions.

In par. 3 a general method of constructing a quadratic form for the free energy of anisotropic media possessing a texture is given.

In par. 4 the theory is illustrated by the case of longitudinal and transverse waves in the texture possessing a conical symmetry ($\infty \cdot m$); dispersion equations connecting the wavelength and frequency are given; some properties of the coefficients of a form which is quadratic in the components and gradients of deformation, are investigated.

1. **1°.** Let a model of a material medium be defined by a finite system of characteristics given by the numbers

$$\mu_1, \ \mu_2, \ldots, \mu_n, \quad k_1, \quad k_2, \ldots, k_m \tag{1.1}$$

where μ_i are the quantities which may be variable, while k_j are quantities constant in the given region of the medium, i.e., physical constants.

In general, the defining parameters are connected by the following differential equations

$$\sum_\alpha A_\alpha (\mu_1, \mu_2, \ldots \mu_n, k_1, k_2, \ldots k_m)\, d\mu_\alpha = 0 \tag{1.2}$$

which can be integrable (holonomic system) or non-integrable (non-holomic system).

Independence of the variable parameters μ_i is by definition based on the fact that virtual displacements $\delta\mu_i$ can, for a given system in a given state, be considered in some region as arbitrary infinitesimals, in particular, as linearly independent quantities.

Defining parameters may include not only such parameters as density, deformation tensor, polarisation vector e.t.c., but also their derivatives with respect to time and space coordinates.

It is known, for example, that the actual strength of some materials depends on the gradient of deformation (see [1]).

In a number of papers the parameters defining the models of media include the spatial derivatives or, more accurately, the gradients of defining parameters. In [1 to 5] defining parameters include deformation gradients and the cases of isotropic media are investigated.

2°. Let the components of some tensor $T_{np} = T_{np}(\xi^1, \xi^2, \xi^3, t)$, and its gradients, be included among the defining parameters. Here ξ^i are the Lagrangian coordinates of a point and t is time.

The gradient of the tensor T_{np} can be considered in the space of initial states, i.e., in the coordinate system fixed with respect to the initial position of the body. We shall denote its components by $\nabla^\circ{}_m T_{np}$. The gradient can be considered in the actual space, i.e. in the coordinate system fixed with respect to the moving body. The latter components shall be denoted by $\nabla\hat{}_m T_{np}$.

We shall use the superscript° to refer to the space of initial states, while the superscript ^ will denote the actual space. Further, we shall assume that T_{np} can be considered as the components of either of the following two tensors $\mathbf{T}^\circ$ or $\mathbf{T}\hat{}$, i.e. $\mathrm{T}^\circ{}_{np} = T\hat{}_{np} = T_{np}$.

Indeed, two tensors $\mathbf{T}^\circ$ and $\mathbf{T}\hat{}$ can be introduced both possessing identical covariant components T_{np} with respect to two different bases

$$\mathbf{T}^\circ = T_{np}\mathfrak{s}^{\circ n}\mathfrak{s}^{\circ p}, \qquad \mathbf{T}\hat{} = T_{np}\mathfrak{s}\hat{}^{\,n}\mathfrak{s}\hat{}^{\,p} \tag{1.3}$$

where $\mathfrak{s}\hat{}^{\,n}$ and $\mathfrak{s}^{\circ n}$ $(n = 1, 2, 3)$ form a contravariant vectorial basis of a Lagrangian coordinate system in the actual space and in the space of initial states, respectively.

The corresponding contravariant components of the above tensors will however be different. The operation of raising the indices of the tensors $\mathbf{T}^\circ$ and $\mathbf{T}\hat{}$ must involve the use of the corresponding tensors $g^{\circ np}$, and $g\hat{}^{\,np}$, the latter being the components of the metric tensors $\mathbf{g}^\circ = g^{\circ np}\mathfrak{s}^\circ{}_n\mathfrak{s}^\circ{}_p$ and $\mathbf{g}\hat{} = g\hat{}^{\,np}\mathfrak{s}\hat{}_n\mathfrak{s}\hat{}_p$ respectively.

We shall show that, in general, the incorporation of $\nabla^\circ{}_m T_{np}$ into the defining parameters is not equivalent to the incorporation of $\nabla\hat{}_m T_{np}$. We have the following familiar formulas

$$\begin{aligned} \nabla\hat{}_m T_{np} &= \frac{\partial T_{np}}{\partial \xi^m} - T_{n\alpha}\Gamma\hat{}^{\,\alpha}_{mp} - T_{\alpha p}\Gamma\hat{}^{\,\alpha}_{mn} \\ \nabla^\circ{}_m T_{np} &= \frac{\partial T_{np}}{\partial \xi^m} - T_{n\alpha}\Gamma^{\circ\alpha}_{mp} - T_{\alpha p}\Gamma^{\circ\alpha}_{mn} \end{aligned} \tag{1.4}$$

Where $\Gamma\hat{}^{\,\alpha}_{mp}$ and $\Gamma^{\circ\alpha}_{mp}$ are the Christoffel symbols

$$\Gamma^{\wedge\alpha}_{mp} = \frac{1}{2} g^{\wedge\alpha s}\left(\frac{\partial g^{\wedge}_{ms}}{\partial\xi^{p}} + \frac{\partial g^{\wedge}_{ps}}{\partial\xi^{m}} - \frac{\partial g^{\wedge}_{mp}}{d\xi^{s}}\right)$$
$$\Gamma^{\circ\alpha}_{mp} = \frac{1}{2} g^{\circ\alpha s}\left(\frac{\partial g^{\circ}_{ms}}{\partial\xi^{p}} + \frac{\partial g^{\circ}_{ps}}{\partial\xi^{m}} - \frac{\partial g^{\circ}_{mp}}{\partial\xi^{s}}\right) \tag{1.5}$$

Let us consider the difference

$$\nabla^{\wedge}_{m}T_{np} - \nabla^{\circ}_{m}T_{np} = T_{n\alpha}(\Gamma^{\circ\alpha}_{mp} - \Gamma^{\wedge\alpha}_{mp}) + T_{\alpha p}(\Gamma^{\circ\alpha}_{mn} - \Gamma^{\wedge\alpha}_{mn}) \tag{1.6}$$

Using the second formula of (1.4) in which $g^{\wedge}_{np}$, replaces T_{np}, we can write the first formula of (1.5), as

$$\Gamma^{\wedge\alpha}_{mp} = \frac{1}{2} g^{\wedge\alpha s}(\nabla^{\circ}_{p}g^{\wedge}_{ms} + g^{\wedge}_{mj}\Gamma^{\circ j}_{ps} + g^{\wedge}_{sj}\Gamma^{\circ j}_{mp} + \nabla^{\circ}_{m}g^{\wedge}_{ps} + g^{\wedge}_{pj}\Gamma^{\circ j}_{ms} +$$
$$+ g^{\wedge}_{sj}\Gamma^{\circ j}_{mp} - \nabla^{\circ}_{s}g^{\wedge}_{mp} - g^{\wedge}_{mj}\Gamma^{\circ j}_{sp} - g^{\wedge}_{pj}\Gamma^{\circ j}_{sm})$$

or

$$\Gamma^{\wedge\alpha}_{mp} = \frac{1}{2} g^{\wedge\alpha s}(\nabla^{\circ}_{p}g^{\wedge}_{ms} + \nabla^{\circ}_{m}g^{\wedge}_{ps} - \nabla^{\circ}_{s}g^{\wedge}_{mp}) + g^{\wedge\alpha s}g^{\wedge}_{sj}\Gamma^{\circ j}_{mp} \tag{1.7}$$

Taking into account the fact that $\nabla^{\circ}_{p}g^{\circ}_{ms} = 0$, we have, from (1.7)

$$\Gamma^{\wedge\alpha}_{mp} - \Gamma^{\circ\alpha}_{mp} = g^{\wedge\alpha s}(\nabla^{\circ}_{p}\varepsilon_{ms} + \nabla^{\circ}_{m}\varepsilon_{ps} - \nabla^{\circ}_{s}\varepsilon_{mp}),\quad \varepsilon_{ms} = \frac{1}{2}(g^{\wedge}_{ms} - g^{\circ}_{ms}) \tag{1.8}$$

Here ε_{ms} are the components of the tensor of finite deformations.

Analogously, we can obtain

$$\Gamma^{\wedge\alpha}_{mp} - \Gamma^{\circ\alpha}_{mp} = g^{\circ\alpha s}(\nabla^{\wedge}_{p}\varepsilon_{ms} + \nabla^{\wedge}_{m}\varepsilon_{ps} - \nabla^{\wedge}_{s}\varepsilon_{mp}) \tag{1.9}$$

Using (1.8) let us write (1.6) as

$$\nabla^{\wedge}_{m}T_{np} = \nabla^{\circ}_{m}T_{np} - T_{n\alpha}g^{\wedge\alpha s}(\nabla^{\circ}_{p}\varepsilon_{ms} + \nabla^{\circ}_{m}\varepsilon_{ps} - \nabla^{\circ}_{s}\varepsilon_{mp}) -$$
$$- T_{\alpha p}g^{\wedge\alpha s}(\nabla^{\circ}_{n}\varepsilon_{ms} + \nabla^{\circ}_{m}\varepsilon_{ns} - \nabla^{\circ}_{s}\varepsilon_{mn}) \tag{1.10}$$

Using

$$L^{*k(ij)}_{mnp} = -T_{n\alpha}g^{\wedge\alpha s}(\delta_{p}{}^{k}\delta_{m}{}^{(i}\delta_{s}{}^{j)} + \delta_{m}{}^{k}\delta_{p}{}^{(i}\delta_{s}{}^{j)} - \delta_{s}{}^{k}\delta_{m}{}^{(i}\delta_{p}{}^{j)}) -$$
$$- T_{\alpha p}g^{\wedge\alpha s}(\delta_{n}{}^{k}\delta_{m}{}^{(i}\delta_{s}{}^{j)} + \delta_{m}{}^{k}\delta_{n}{}^{(i}\delta_{s}{}^{j)} - \delta_{s}{}^{k}\delta_{m}{}^{(i}\delta_{n}{}^{j)}) = L^{*k(ij)}_{mnp}(T_{\gamma\rho}, g^{\wedge ab}) \tag{1.11}$$

we obtain, from (1.10)

$$\nabla^{\wedge}_{m}T_{np} = \nabla^{\circ}_{m}T_{np} + L^{*k(ij)}_{mnp}\nabla^{\circ}_{k}\varepsilon_{ij} \tag{1.12}$$

Here the round parentheses enclosing the indices (ij) define the operation of symmetrising the tensor with respect to the corresponding indices, and dividing the result by two.

Replacing T_{np} with ε_{np}, yields

$$\nabla^{\wedge}{}_{m}\varepsilon_{np}=L_{m\,(np)}^{k\,(ij)}\nabla^{\circ}{}_{k}\varepsilon_{ij},\quad L_{m\,(np)}^{k\,(ij)}=\delta_{m}{}^{k}\delta_{(n}{}^{(i}\delta_{p)}{}^{j)}+L^{*k\,(ij)}_{m\,(np)}\,(\varepsilon_{\gamma\rho},\ g^{\wedge ab}) \tag{1.13}$$

i.e. the conclusion is reached that, from the theoretical point of view, it is immaterial whether $\nabla^{\wedge}{}_{m}\,\varepsilon_{np}$ or $\nabla^{\circ}{}_{m}\varepsilon_{np}$, are included in the defining parameters, since one of them can be expressed in terms of the other according to the formula (1.13).

If the deformations are small enough to make the terms $\varepsilon_{\alpha\beta}\nabla^{\circ}{}_{k}\varepsilon_{ij}$ negligible, then we shall have

$$\nabla^{\wedge}{}_{m}\varepsilon_{np}=\nabla^{\circ}{}_{m}\varepsilon_{np} \tag{1.14}$$

with the accuracy of up to the infinitesimals of the higher order or, in other words, for small deformations the above gradients coincide.

A formula of the type (1.12) can be written for a tensor of any rank l

$$\nabla^{\wedge}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l}=\nabla^{\circ}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l}+L^{*k\,(ij)}_{m\beta_1,\beta_2,\ldots,\beta_l}\nabla^{\circ}{}_{k}\varepsilon_{ij} \tag{1.15}$$

from which we see that the inclusion of $\nabla^{\wedge}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l}$ among the defining parameters is not equivalent to the inclusion of $\nabla^{\circ}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l}$, unless the latter is supplemented with $\nabla^{\circ}{}_{k}\varepsilon_{ij}$. Hence, (1.15) gives us the relationship between two sets of the defining parameters.

If the deformations are small and the condition

$$\frac{T_{\beta_1,\beta_2,\ldots,\beta_l}}{\nabla^{\circ}{}_{k}T_{\alpha_1,\alpha_2,\ldots,\alpha_l}}\nabla^{\circ}{}_{a}\varepsilon_{bc}\ll 1$$

is satisfied, then

$$\nabla^{\wedge}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l}=\nabla^{\circ}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l} \tag{1.16}$$

If μ is a scalar, then we always have

$$\nabla^{\wedge}{}_{k}\mu=\nabla^{\circ}{}_{k}\mu \tag{1.17}$$

Substitution of

$$\nabla^{\circ}{}_{k}\varepsilon_{ij}=L^{-1m\,(np)}_{k\,(ij)}\nabla^{\wedge}{}_{m}\varepsilon_{np} \tag{1.18}$$

into (1.15) results in the expression for $\nabla^{\circ}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l}$ in terms of

$$\nabla^{\wedge}{}_{m}T_{\beta_1,\beta_2,\ldots,\beta_l},\qquad T_{\beta_1,\beta_2,\ldots,\beta_l},\qquad \varepsilon_{ij},\qquad \nabla^{\wedge}{}_{m}\varepsilon_{np}$$

2. Let us make the following assumptions.

1°. In the following, only reversible processes will be considered, although all the assumptions will remain in force if only we assume that

$$dQ^{(e)}=TdS \tag{2.1}$$

where T is the temperature, S is the entropy and $dQ^{(e)}$ is the increase in the heat energy.

2°. Let us consider the free energy F

$$F = F(T,\ g^{\circ ij},\ \varepsilon_{ij},\ \ \nabla^{\wedge}{}_k\varepsilon_{ij},\ n^{\wedge i},\ \ \nabla^{\wedge}{}_k n^{\wedge i}) = F(\mu_i,\ k_j) \tag{2.2}$$

Here ε_{ij} is the tensor of finite deformations and n^i is a vector.

We assume, that the metric tensor in the space of initial states $g^{\circ ij}$ is independent of time

$$g^{\circ ij} = g^{\circ ij}(\xi^1,\ \xi^2,\ \xi^3)$$

The free energy increment for the elementary particle is, by the 2-*nd* Law of Thermodynamics

$$dF = \frac{\partial F}{\partial T}dT + \frac{\partial F}{\partial \varepsilon_{ij}}d\varepsilon_{ij} + \frac{\partial F}{\partial \nabla^{\wedge}{}_m\varepsilon_{np}}d\nabla^{\wedge}{}_m\varepsilon_{np} + \frac{\partial F}{\partial n^{\wedge i}}dn^{\wedge i} + $$
$$+ \frac{\partial F}{\partial \nabla^{\wedge}{}_k n^{\wedge i}}d\nabla^{\wedge}{}_k n^{\wedge i} = \frac{p^{\wedge ij}}{\rho}e_{ij}\,dt - S\,dT + dq^{**} \tag{2.3}$$

where e_{ij} is the tensor of the velocities of deformation. If $g^{\circ ij}$ is independent of time, then $e_{\alpha\beta}dt = d\varepsilon_{\alpha\beta}$.

3°. We shall assume that dq^{**} is the additional energy flux through the surface Σ surrounding the given element of volume V of mass dm, i.e.

$$dq^{**}\,dm = \int_{\Sigma} S^k n_k\,d\sigma\,dt = \int_{V} \nabla^{\wedge}{}_k S^k\,d\tau\,dt \tag{2.4}$$

here n_k are the components of the normal to Σ. The fact that V is small implies that the integral in (2.4) can be replaced by its integrand; taking into account the fact that $dm = \rho_0 d\tau_0 = \rho d\tau$, we have

$$dq^{**} = \frac{\nabla^{\wedge}{}_k S^k\,d\tau\,dt}{dm} = \frac{1}{\rho}\nabla^{\wedge}{}_k S^k\,dt \tag{2.5}$$

4°. We assume that the energy flux dq^{**} is proportional to the increase in the defining parameters, i.e.

$$S^k dt = Q^{kij}d\varepsilon_{ij} + R^{klij}\,d\nabla^{\wedge}{}_l\varepsilon_{ij} + L^k{}_i dn^{\wedge i} + m^{kl}{}_i d\nabla^{\wedge}{}_l n^{\wedge i} + A^k dT \tag{2.6}$$

where Q^{kij}, R^{klij}, $L^k{}_i$, $m^{kl}{}_i$, and A^k also depend on the parameters (2.2).

5°. We assume that there are no non-holonomic relationships between the defining parameters and that $d\mu_i$ together with $\nabla^{\wedge}{}_k d\mu_i$ can be considered independent. Let us use the formulas

$$d\nabla^{\wedge}{}_m\varepsilon_{np} = L^{k\,(ij)}_{m\,(np)}\nabla^{\wedge}{}_k d\varepsilon_{ij},\quad d\nabla^{\wedge}{}_m n^{\wedge p} = \nabla^{\wedge}{}_m dn^{\wedge p} + \psi^{k\,(ij)p}_m\nabla^{\wedge}{}_k d\varepsilon_{ij} \tag{2.7}$$

the derivation of which will be given in the appendix A. Here

$$L^{k\,(ij)}_{m\,(np)} = \delta^k_m\delta^{(i}_{(n}\delta^{j)}_{p)} - \varepsilon_{n\alpha}g^{\wedge\alpha s}\,[\delta^k_p\delta_m{}^{(i}\delta_s{}^{j)} + \delta_m{}^k\delta_p{}^{(i}\delta_s{}^{j)} - \delta_s{}^k\delta_m{}^{(i}\delta_p{}^{j)}] -$$
$$- \varepsilon_{\alpha p}g^{\wedge\alpha s}\,[\delta_n{}^k\delta_m{}^{(i}\delta_s{}^{j)} + \delta_m{}^k\delta_n{}^{(i}\delta_s{}^{j)} - \delta_s{}^k\delta_m{}^{(i}\delta_n{}^{j)}] \tag{2.8}$$

$$\psi_m^{k\,(ij)\,p} = \tfrac{1}{2}\,\delta_m{}^k\,(n^{\wedge i}g^{\wedge pj} + n^{\wedge j}g^{\wedge pi}) + \tfrac{1}{2}\,n^{\wedge k}\,(g^{\wedge pj}\delta_m{}^i + g^{\wedge pi}\delta_m{}^j) -$$
$$-\tfrac{1}{2}\,g^{\wedge pk}\,(n^{\wedge i}\delta_m{}^j + n^{\wedge j}\delta_m{}^i)$$

Comparing the coefficients in the left- and right-hand sides of the equation (2.3) we have, for independent increments

$$A^k = 0, \quad m^{kl}{}_i = 0, \quad R^{klij} = 0, \quad \frac{\partial F}{\partial T} = -S \tag{2.9}$$

$$\frac{\partial F}{\partial \nabla^{\wedge}{}_k n^{\wedge i}} = \frac{1}{\rho} L^k{}_i, \quad \frac{\partial F}{\partial n^{\wedge i}} = \frac{1}{\rho}\,\nabla^{\wedge}{}_k L^k{}_i \tag{2.10}$$

$$p^{\wedge ij} = \rho\,\frac{\partial F}{\partial \varepsilon_{ij}} - \nabla^{\wedge}{}_k Q^{kij}$$

$$\frac{\partial F}{\partial \nabla^{\wedge}{}_m \varepsilon_{np}}\, L_{m\,(np)}^{k\,(ij)} + \frac{\partial F}{\partial \nabla^{\wedge}{}_m n^{\wedge p}}\, \psi_m^{k\,(ij)\,p} = \frac{1}{\rho}\, Q^{kij} \tag{2.11}$$

which represent the equations of motion, the continuity equation and the law of conservation of energy, and from a closed system of equations for the components of the vector $\mathbf{n} = n^{\wedge i}\partial^{\wedge}{}_i$, for the displacement vector $\mathbf{u} = u^{\wedge i}\partial^{\wedge}{}_i$, for the temperature T and the entropy S.

Eliminating $L^k{}_i$ from (2.10), we obtain

$$\frac{\partial F}{\partial n^{\wedge i}} = \frac{1}{\rho}\,\nabla^{\wedge}{}_k\left(\rho\,\frac{\partial F}{\partial \nabla^{\wedge}{}_k n^{\wedge i}}\right) \tag{2.12}$$

which, can be used to determine $n^{\wedge i}$ in terms of the other defining parameters, when the relationship between F and the defining parameters is known.

The stresses are determined from (2.11.1) where Q^{kij} given by (2.11.2) is used. Equation (2.9.4) is used to determine the entropy S.

Assuming that the energy gain dq^{**} exists, we have

$$dq^{**} = \frac{1}{\rho}\,\nabla^{\wedge}{}_k\,[Q^{kij}\,d\varepsilon_{ij} + L^k{}_i\,dn^{\wedge i}] = \tag{2.13}$$
$$= \frac{1}{\rho}\,\nabla^{\wedge}{}_k\left[\rho\left(\frac{\partial F}{\partial \nabla^{\wedge}{}_m \varepsilon_{np}}\, L_{m\,(np)}^{k\,(ij)} + \frac{\partial F}{\partial \nabla^{\wedge}{}_m n^{\wedge p}}\,\psi_m^{k\,(ij)\,p}\right) d\varepsilon_{ij} + \rho\,\frac{\partial F}{\partial \nabla^{\wedge}{}_k n^{\wedge i}}\, dn^{\wedge i}\right]$$

If $\nabla^{\wedge}{}_k n^{\wedge i}$ are not included in the defining parameters, we have

$$\frac{\partial F}{\partial n^{\wedge i}}\, dn^{\wedge i} = 0, \qquad L^k{}_i = 0 \tag{2.14}$$

If $n^{\wedge i}$ belongs to the parameters of the type μ_i, then since $dn^{\wedge i} \neq 0$, (2.14) will give

$$\partial F \,/\, \partial n^{\wedge i} = 0 \tag{2.15}$$

which can be regarded as conditions for the determination of $n^{\wedge i}$ in terms of known F.

If $n^{\wedge i}$ is one of the parameters of the type k_j, i.e. $n^{\wedge i} = n^{\wedge i}\,(\xi^1, \xi^2, \xi^3)$, then for a particle $dn^{\wedge i} = 0$ and Equations (2.15) do not follow. In this case $n^{\wedge i}$ should be assigned to each particle as a constant external parameter. As an example let $n^{\wedge i}$ define the anisotropy of a material medium possessing a definite texture and let the anisotropy of each particle be constant with respect to time. This will be precisely the case of $n^{\wedge i}$ belonging to the set of parameters of the type k_j. The energy gain dq^{**} will, in this case, be given by the following expression

$$dq^{**} = \frac{1}{\rho} \nabla^{\wedge}{}_k [Q^{kij} d\varepsilon_{ij}] = \frac{1}{\rho} \nabla^{\wedge}{}_k \left[\rho \frac{\partial F}{\partial \nabla^{\wedge}{}_m \varepsilon_{np}} L^{k\,(ij)}_{m\,(np)}\, d\varepsilon_{ij}\right] \tag{2.16}$$

which makes it clear that the energy exchange dq^{**} between the particles takes place only, when the deformations are time-dependent, i.e. when $d\varepsilon_{ij} \neq 0$. In the static case, $dq^{**} = 0$.

If on the other hand $\nabla^{\wedge}{}_k n^{\wedge i}$, are included amongst the defining parameters then, even in the static case, the energy flux dq^{**} into the particle differs from zero, if is time-dependent, i.e. if $dn^{\wedge i} \neq 0$.

3. 1°. In case of small deformations the theory of elasticity gives the following quadratic form for the free energy

$$F = A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} \tag{3.1}$$

or, for the isotropic medium

$$2F = \lambda\, (\varepsilon_{ll})^2 + 2\mu \varepsilon_{ik} \varepsilon_{ik} \tag{3.2}$$

where λ, and μ are Lamé parameters.

We shall assume that in case when $\nabla^{\wedge}{}_k\, \varepsilon_{ij}$, are, together with ε_{ij}, included in the defining parameters, then the free energy will also be represented as a quadratic form

$$F = A^{ijkl} \varepsilon_{ij} \varepsilon_{kl} + B^{ijklm} \varepsilon_{ij} \nabla^{\wedge}{}_m \varepsilon_{kl} + C^{ijklmn} \nabla^{\wedge}{}_m \varepsilon_{ij} \nabla^{\wedge}{}_n \varepsilon_{kl} \tag{3.3}$$

Here the tensors A^{ijkl}, B^{ijklm}, and C^{ijklmn} are symmetric with respect to the indices ij and kl. The tensor A^{ijkl} is also symmetric with respect to the interchange of pairs of indices ij and kl, while the tensor C^{ijklmn} is also symmetric with respect to the simulataneous interchange of the indices i and k, j and l, m and n.

In the case of small deformations $\nabla^{\wedge}{}_k \varepsilon_{ij}$ can be replaced by $\nabla^{\circ}{}_k \varepsilon_{ij}$; or, in the cartesian coordinate system, simply

$$\frac{\partial \varepsilon_{ij}}{\partial x^k} = \varepsilon_{ij,\,k} \tag{3.4}$$

If the material medium possesses a definite symmetry group, then the tensors A^{ijkl}, B^{ijklm} and C^{ijklmn} will be invariant with respect to this symmetry group.

General form of sych tensors up to the fourth rank and for any symmetry group, is given in [6]. The author has at his disposal general forms of tensors of the fifth and sixth rank invariant with respect to all the seven texture configurations. They are omitted from here because of their bulk. They are of the type $T = k_1 \mathrm{T}_1 + \ldots + k_p \mathrm{T}_p$, where T_n are the linearly independent tensors formed from the tensors defining the symmetry group of each particular texture, while k_n are scalar functions. Thereby, the number p is, for tensors of the rank $r = 5$ and $r = 6$, equal to

	$\infty/\infty \cdot m$	∞/∞	$\infty \cdot m$	$m \cdot \infty : m$	$\infty:2$	∞	$\infty:m$	
$p =$	0	6	26	0	25	51	0	$(r = 5)$
$p =$	15	15	71	71	71	141	141	$(r = 6)$

(symbols for the types of texture are taken from [6]).

2°. Consider for example a material medium possessing a symmetry group $(\infty \cdot m)$ and given by the metric tensor $\mathbf{g} = g^{ij}\partial_i\partial_j$, together with the anisotropy vector $\mathbf{n} = n^i\partial_i$. The free energy will, in this case, have the following form

$$\begin{aligned}
F = {} & k_1 (\varepsilon_{ii})^2 + k_2\varepsilon_{ij}\varepsilon_{ij} + k_3 n_i n_j \varepsilon_{kk}\varepsilon_{ij} + k_4 n_i n_j \varepsilon_{ik}\varepsilon_{jk} + k_5 n_i n_j n_k n_l \varepsilon_{ij}\varepsilon_{kl} + \\
& + n_i [k_6\varepsilon_{ij}\varepsilon_{jl,l} + k_7\varepsilon_{ij}\varepsilon_{ll,j} + k_8\varepsilon_{jj}\varepsilon_{il,l} + k_9\varepsilon_{jl}\varepsilon_{il,j} + k_{10}\varepsilon_{jj}\varepsilon_{ll,i} + k_{11}\varepsilon_{jl}\varepsilon_{jl,i}] + \\
& + n_i n_j n_l [k_{12}\varepsilon_{ij}\varepsilon_{lk,k} + k_{13}\varepsilon_{ij}\varepsilon_{kk,l} + k_{14}\varepsilon_{kk}\varepsilon_{ij,l} + k_{15}\varepsilon_{ik}\varepsilon_{jl,k} + k_{16}\varepsilon_{ik}\varepsilon_{kj,l}] + \\
& + k_{17} n_i n_j n_k n_l n_m \varepsilon_{ij}\varepsilon_{lk,m} + k_{18}\varepsilon_{ij,k}\varepsilon_{ij,k} + k_{19}\varepsilon_{ij,k}\varepsilon_{ik,j} + k_{20}\varepsilon_{ij,i}\varepsilon_{kj,k} + \\
& + k_{21}\varepsilon_{ij,i}\varepsilon_{kk,j} + k_{22}\varepsilon_{ii,j}\varepsilon_{kk,j} + n_i n_j [k_{23}\varepsilon_{ij,k}\varepsilon_{kl,l} + k_{24}\varepsilon_{ij,k}\varepsilon_{ll,k} + k_{25}\varepsilon_{il,j}\varepsilon_{lk,k} + \\
& + k_{26}\varepsilon_{il,j}\varepsilon_{kk,l} + k_{27}\varepsilon_{il,k}\varepsilon_{jl,k} + k_{28}\varepsilon_{il,k}\varepsilon_{jk,l} + k_{29}\varepsilon_{il,l}\varepsilon_{jk,k} + k_{30}\varepsilon_{il,k}\varepsilon_{lk,j} + \\
& + k_{31}\varepsilon_{il,l}\varepsilon_{kk,j} + k_{32}\varepsilon_{ll,i}\varepsilon_{kk,j} + k_{33}\varepsilon_{lk,i}\varepsilon_{lk,j}] + n_i n_j n_k n_l [k_{34}\varepsilon_{ij,k}\varepsilon_{lm,m} + \\
& + k_{35}\varepsilon_{ij,k}\varepsilon_{mm,l} + k_{36}\varepsilon_{ij,m}\varepsilon_{kl,m} + k_{37}\varepsilon_{ij,m}\varepsilon_{mk,l} + k_{38}\varepsilon_{im,j}\varepsilon_{mk,l}] + \\
& + k_{39} n_i n_j n_k n_l n_m n_p \varepsilon_{ij,k}\varepsilon_{lm,p}
\end{aligned} \tag{3.5}$$

Hence, the model of the considered medium is fully defined by 39 physical constants amongst which $2k_1 = \lambda$, and $k_2 = \mu$ where λ and μ are Lamé parameters. Coefficients $k_1, \ldots, k_{39}$ may depend on the temperature and on the modulus of the anisotropy vector. For small deformations, we have

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right) \tag{3.6}$$

Equations of motion then become

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \rho_0 F_i + \frac{\partial p_{ij}}{\partial x_j} = \rho_0 F_i + \rho_0 \frac{\partial}{\partial x_j}\left[\frac{\partial F}{\partial \varepsilon_{ij}} - \frac{\partial}{\partial x_k}\left(\frac{\partial F}{\partial \nabla_k \varepsilon_{ij}}\right)\right] \tag{3.7}$$

where F_i are the external mass forces;

$$\begin{aligned}
p_{ij} = {} & \frac{1}{2}\rho_0 \{2 [2\varepsilon_{kk}\delta_{ij}k_1 + 2k_2\varepsilon_{ij} + k_3 (n_i n_j \varepsilon_{kk} + n_k n_l \varepsilon_{kl}\delta_{ij}) + k_4 n_k (n_i\varepsilon_{kj} + n_j\varepsilon_{ki}) + \\
& + 2k_5 n_i n_j n_l n_m \varepsilon_{lm}] + k_6 (n_i\varepsilon_{jl,l} + n_j\varepsilon_{il,l}) + k_7 (n_i\varepsilon_{ll,j} + n_j\varepsilon_{ll,i}) + 2k_8 n_k \delta_{ij}\varepsilon_{kl,l} + \\
& + k_9 n_k (\varepsilon_{ki,j} + \varepsilon_{kj,i}) + 2k_{10} n_k \delta_{ij}\varepsilon_{ll,k} + 2k_{11} n_k \varepsilon_{ij,k} + 2k_{12} n_i n_j n_l \varepsilon_{lk,k} + \\
& + 2k_{13} n_i n_j n_l \varepsilon_{kk,l} + 2k_{14}\delta_{ij} n_k n_l n_m \varepsilon_{kl,m} + k_{15} n_k n_l (n_i\varepsilon_{kl,j} + n_j\varepsilon_{kl,i}) + \\
& + k_{16} n_k n_l (n_i\varepsilon_{jk,l} + n_j\varepsilon_{ik,l}) + 2k_{17} n_i n_j n_k n_l n_m \varepsilon_{lk,m}\} - {}^1\!/_2\, \rho_0 \frac{\partial}{\partial x_k} \{k_6 n_l (\delta_{ik}\varepsilon_{lj} + \delta_{jk}\varepsilon_{li}) + \\
& + 2k_7 n_l \delta_{ij}\varepsilon_{lk} + k_8 \varepsilon_{ll} (n_i\delta_{jk} + n_j\delta_{ik}) + k_9 (n_i\varepsilon_{jk} + n_j\varepsilon_{ik}) + 2k_{10} n_k \delta_{ij}\varepsilon_{ll} + 2k_{11} n_k \varepsilon_{ij} + \\
& + k_{12} n_l n_m \varepsilon_{lm} (n_i\delta_{jk} + n_j\delta_{ik}) + 2k_{13} n_k n_l n_m \delta_{ij}\varepsilon_{lm} + 2k_{14} n_i n_j n_k \varepsilon_{ll} + 2k_{15} n_i n_j n_l \varepsilon_{kl} + \\
& + k_{16} n_k n_l (n_j\varepsilon_{il} + n_i\varepsilon_{jl}) + 2k_{17} n_i n_j n_k n_l n_m \varepsilon_{lm} + 4k_{18}\varepsilon_{ij,k} + 2k_{19} (\varepsilon_{ik,j} + \varepsilon_{jk,i}) +
\end{aligned} \tag{3.8}$$

$$+ 2k_{20}(\delta_{ik}\varepsilon_{lj,l} + \delta_{jk}\varepsilon_{li,l}) + k_{21}(\delta_{ik}\varepsilon_{ll,j} + \delta_{jk}\varepsilon_{ll,i} + 2\delta_{ij}\varepsilon_{lk,l}) + 4k_{22}\delta_{ij}\varepsilon_{ll,k} +$$
$$+ k_{23}[2n_in_j\varepsilon_{kl,l} + n_ln_m(\varepsilon_{lm,i}\delta_{jk} + \varepsilon_{lm,j}\delta_{ik})] + 2k_{24}(n_in_j\varepsilon_{ll,k} + \delta_{ij}n_ln_m\varepsilon_{lm,k}) +$$
$$+ k_{25}[n_k(n_i\varepsilon_{jl,l} + n_j\varepsilon_{il,l}) + n_ln_m(\delta_{jk}\varepsilon_{il,m} + \delta_{ik}\varepsilon_{jl,m})] + k_{26}[n_k(n_i\varepsilon_{ll,j} + n_j\varepsilon_{ll,i}) +$$
$$+ 2n_ln_m\delta_{ij}\varepsilon_{lk,m}] + 2k_{27}(n_in_l\varepsilon_{lj,k} + n_jn_l\varepsilon_{li,k}) + 2k_{28}n_l(n_i\varepsilon_{lk,j} + n_j\varepsilon_{lk,i}) +$$
$$+ 2k_{29}n_l\varepsilon_{lm,m}(n_i\delta_{jk} + n_j\delta_{ik}) + k_{30}n_l[n_i\varepsilon_{jk,l} + n_j\varepsilon_{ik,l} + n_k(\varepsilon_{li,j} + \varepsilon_{lj,i})] +$$
$$+ k_{31}n_l[\varepsilon_{mm,l}(n_i\delta_{jk} + n_j\delta_{ik}) + 2n_k\delta_{ij}\varepsilon_{lm,m}] + 4k_{32}n_kn_l\delta_{ij}\varepsilon_{mm,l} + 4k_{33}n_kn_l\varepsilon_{ij,l} +$$
$$+ k_{34}[2n_in_jn_kn_l\varepsilon_{lm,m} + n_ln_mn_p\varepsilon_{lm,p}(n_i\delta_{jk} + n_j\delta_{ik})] + 2k_{35}n_kn_l(n_in_j\varepsilon_{mm,l} +$$
$$+ n_mn_p\delta_{ij}\varepsilon_{lm,p}) + 4k_{36}n_in_jn_ln_m\varepsilon_{lm,k} + k_{37}n_ln_m[2n_in_j\varepsilon_{kl,m} + n_k(n_j\varepsilon_{lm,i} +$$
$$+ n_i\varepsilon_{lm,j})] + 2k_{38}n_kn_ln_m(n_i\varepsilon_{jl,m} + n_j\varepsilon_{il,m}) + 4k_{39}n_in_jn_kn_ln_mn_p\varepsilon_{lm,p}\}$$

$$\delta_{ij} = 1 \quad \text{for} \quad i = j, \qquad \delta_{ij} = 0 \quad \text{for} \quad i \neq j$$

From the continuity equation we have, for small deformations

$$\rho = \rho_0\left(1 - \frac{\partial u_\alpha}{\partial x_\alpha}\right) \tag{3.9}$$

where u_α are the components of the displacement vector.

4. 1°. Let us assume, in addition, that the external mass forces are absent, that $\mathbf{n}$ is time and coordinate independent and let us consider the case of longitudinal waves

$$u_1 = u = u(x), \qquad u_2 = u_3 \equiv 0 \tag{4.1}$$

Then, the equations of motion reduce to a single equation

$$\frac{\partial^2 u}{\partial t^2} = A\frac{\partial^2 u}{\partial x^2} - B\frac{\partial^4 u}{\partial x^4} \tag{4.2}$$

$$A = 2k_1 + 2k_2 + 2k_3(n_1)^2 + 2k_4(n_1)^2 + 2k_5(n_1)^4 \tag{4.3}$$

$$\begin{aligned} B = {} & 2(k_{18} + k_{19} + k_{20} + k_{21} + k_{22}) + 2(n_1)^2(k_{23} + \\ & + k_{24} + k_{25} + k_{26} + k_{27} + k_{28} + k_{29} + k_{30} + k_{31} + k_{32} + \\ & + k_{33}) + 2(n_1)^4(k_{34} + k_{35} + k_{36} + k_{37} + k_{38}) + 2(n_1)^6 k_{39} \end{aligned} \tag{4.4}$$

From the conditions of stability we preuppose that $k_i \geqslant 0$. Then $A > 0$ and $B \geqslant 0$. We shall seek the solution of (4.2) in the form

$$u = f(x - at) \tag{4.5}$$

Solving (4.2) we obtain

$$u = c_1\cos\left[\left(\frac{a^2 - A}{B}\right)^{1/2}(x - at)\right] + c_2\sin\left[\left(\frac{a^2 - A}{B}\right)^{1/2}(x - at)\right] \tag{4.6}$$

$$a^2 = A + \frac{(2\pi)^2 B}{\lambda^2}, \quad \lambda = 2\pi\left(\frac{B}{a^2 - A}\right)^{1/2}, \quad T = \frac{2\pi}{a}\left(\frac{B}{a^2 - A}\right)^{1/2} = \frac{\lambda^2}{\sqrt{A\lambda^2 + 4\pi^2 B}} \tag{4.7}$$

Here a is the velocity of the wave propagation, λ is the wavelength and T is the period.

Since $A > 0$ and $B \geqslant 0$, we see from (4.7) that in the given medium, the velocity of perturbation a, can never assume the value smaller than that of the velocity of perturbation in the Hooke's elastic medium of the theory of elasticity, and approaches it at large values of λ.

When λ is small, then from (4.7) it follows that

$$a^2 = 4\pi^2 B / \lambda^2, \qquad T = \lambda^2 / 2\pi \sqrt{B} \tag{4.8}$$

which means that both, a and T are defined in terms of the coefficient B, which in turn is defined by the coefficients of the quadratic form, which multiply the deformation gradients.

2°. Under the assumptions concerning n_i and F_i which were already given in par. 4.1°, we consider the case of transverse waves

$$u_2 = u_2(x_1) = u_2(x), \qquad u_3 = u_3(x_1) = u_3(x), \qquad u_1 \equiv 0 \tag{4.9}$$

Since n_i are coordinate-independent we can, after the transformation of coordinates x_2 and x_3 (rotation in the plane $x_2 x_3$), select a coordinate system in which $n_3 = 0$. In this system the equations of motion will be given by

$$\begin{gathered} 0 = \frac{\partial p_{11}}{\partial x} = \frac{\partial}{\partial x}\left(l_1 \frac{\partial u_2}{\partial x} + l_2 \frac{\partial^2 u_2}{\partial x^2} + l_3 \frac{\partial^3 u_2}{\partial x^3}\right) \\ \frac{\partial^2 u_2}{\partial t^2} = \alpha \frac{\partial^2 u_2}{\partial x^2} - A_1 \frac{\partial^4 u_2}{\partial x^4}, \qquad \frac{\partial^2 u_3}{\partial t^2} = \beta \frac{\partial^2 u_3}{\partial x^2} - B_1 \frac{\partial^4 u_3}{\partial x^4} \end{gathered} \tag{4.10}$$

$$\begin{aligned} \alpha &= 4k_2 + 2k_4 (n_2^2 + n_1^2) + 4k_5 (n_1 n_2)^2 \\ A_1 &= 4k_{18} + 2k_{19} + 2k_{20} + 2k_{25} n_1^2 + k_{27} (n_1^2 + n_2^2) + \\ &\quad + 2k_{28} n_2^2 + 2k_{29} n_2^2 + 2k_{30} n_1^2 + 4k_{33} n_1^2 \\ \beta &= 4k_2 + 2k_4 n_1^2 \\ B_1 &= 4k_{18} + 2k_{19} + 2k_{20} + n_1^2 (2k_{25} + k_{27} + 2k_{30} + 4k_{33}) \\ l_1 &= 4k_3 n_1 n_2 + 4k_4 n_1 n_2 + 4k_5 n_1^3 n_2 \\ l_2 &= 2n_2 (k_8 + k_9) - 2n_2 (k_6 + k_7) + n_1^2 n_2 (-2k_{12} + 4k_{14} - 4k_{13} + 2k_{15}) \\ -l_3 &= 2n_1 n_2 (2k_{23} + 2k_{24} + k_{25} + k_{26} + 2k_{27} + 2k_{28} + 2k_{29} + \\ &\quad + 2k_{30} + k_{31}) + 2n_1^3 n_2 (3k_{34} + 2k_{35} + 4k_{36} + 3k_{37} + 4k_{38}) + 8k_{39} n_1^5 n_2 \end{aligned} \tag{4.11}$$

We shall seek the solution of (4.10) in the form

$$u_2 = f_2(x - bt), \quad u_3 = (x - at)$$

Then, if we assume that $n_2 \neq 0$, the solution will be

$$u_2 \equiv 0, \qquad u_3 = c_1 \cos\left[\left(\frac{a^2 - \beta}{B_1}\right)^{1/2} (x - at)\right] + c_2 \sin\left[\left(\frac{a^2 - \beta}{B_1}\right)^{1/2} (x - at)\right] \tag{4.12}$$

from which we can obtain the relations analogous to (4.7)

$$a^2 = \beta + \frac{4\pi^2 B_1}{\lambda^2}, \qquad \lambda = 2\pi \left(\frac{B_1}{a^2 - \beta}\right)^{1/2}, \quad T = \frac{2\pi}{a}\left(\frac{B_1}{a^2 - \beta}\right)^{1/2} = \frac{\lambda^2}{\sqrt{\beta\lambda^2 + 4\pi^2 B_1}} \tag{4.13}$$

From (4.13) it follows that the plane of polarisation of the transverse wave is perpendicular to the plane formed by the direction of the anisotropy vector **n** and the direction of the wave front. If $n_2 = n_3 = 0$, then the plane of polarisation can be arbitrary.

Note. In both, longitudinal and transverse cases we obtain the relationship between the wavelength λ and its frequency $\omega = 2\pi/T$, in the form of dispersion equations

$$\omega_{-} = \frac{2\pi}{\lambda^2}\sqrt{A\lambda^2 + 4\pi^2 B}, \qquad \omega_{\perp} = \frac{2\pi}{\lambda^2}\sqrt{\beta\lambda^2 + 4\pi^2 B_1} \tag{4.14}$$

From (4.14) it follows that for large wavelengths, the frequency is related to the wavelength in the manner as in Hooke's media

$$\omega_{-} = 2\pi\sqrt{A}\,/\,\lambda, \qquad \omega_{\perp} = 2\pi\sqrt{\beta}\,/\,\lambda$$

while in case of small wavelengths, the frequency is determined from the coefficients appearing in the quadratic form in front of the deformation gradients, according to the formulas

$$\omega_{-} = 4\pi^2\sqrt{B}\,/\,\lambda^2, \qquad \omega_{\perp} = 4\pi^2\sqrt{B_1}\,/\,\lambda^2 \tag{4.15}$$

The velocity of propagation varies with wavelength and is calculated by means of (4.13) for transverse waves, and by means of (4.7) for longitudinal waves.

If the direction of wave propagation is orthogonal to the anisotropy vector then, since in this case A, B, β, and B_1 are independent of n_i, it follows, that the anisotropy of the medium does not influence the frequency, period or the velocity of propagation and, that the waves propagates just as they do in the isotropic medium.

Appendix *A*. Derivation of the formulas (2.7). We have the well known formulas

$$\begin{aligned} \nabla\hat{}_m d\varepsilon_{np} &= \frac{\partial d\varepsilon_{np}}{\partial\xi^m} - d\varepsilon_{n\alpha}\Gamma\hat{}^{\alpha}_{mp} - d\varepsilon_{\alpha p}\Gamma\hat{}^{\alpha}_{mn} \\ d\nabla\hat{}_m\varepsilon_{np} &= d\left(\frac{\partial\varepsilon_{np}}{\partial\xi^m} - \varepsilon_{n\alpha}\Gamma\hat{}^{\alpha}_{mp} - \varepsilon_{\alpha p}\Gamma\hat{}^{\alpha}_{mn}\right) \end{aligned} \tag{A.1}$$

Equating them, we obtain

$$d\nabla\hat{}_m\varepsilon_{np} = \nabla\hat{}_m d\varepsilon_{np} - \varepsilon_{n\alpha}d\Gamma\hat{}^{\alpha}_{mp} - \varepsilon_{\alpha p}d\Gamma\hat{}^{\alpha}_{mn} \tag{A.2}$$

$$\Gamma\hat{}^{\alpha}_{mp} = \frac{1}{2}g\hat{}^{\alpha s}\left(\frac{\partial g\hat{}_{ms}}{\partial\xi^p} + \frac{\partial g\hat{}_{ps}}{\partial\xi^m} - \frac{\partial g\hat{}_{mp}}{\partial\xi^s}\right) \tag{A.3}$$

$$\begin{aligned} d\Gamma\hat{}^{\alpha}_{mp} &= \frac{1}{2}\left(\frac{\partial g\hat{}_{ms}}{\partial\xi^p} + \frac{\partial g\hat{}_{ps}}{\partial\xi^m} - \frac{\partial g\hat{}_{mp}}{\partial\xi^s}\right)dg\hat{}^{\alpha s} + \\ &+ \frac{1}{2}g\hat{}^{\alpha s}\left(\frac{\partial dg\hat{}_{ms}}{\partial\xi^p} + \frac{\partial dg\hat{}_{ps}}{\partial\xi^m} - \frac{\partial dg\hat{}_{mp}}{\partial\xi^s}\right) \\ 2\Gamma\hat{}^{j}_{mp}g\hat{}_{js} &= \frac{\partial g\hat{}_{ms}}{\partial\xi^p} + \frac{\partial g\hat{}_{ps}}{\partial\xi^m} - \frac{\partial g\hat{}_{mp}}{\partial\xi^s} \end{aligned} \tag{A.4}$$

Using the second formula of (A.4) and the first formula of (A.1) in which $d\varepsilon_{np} = {}^1/_2 d\,(g\hat{}_{np} - g^{\circ}_{np})$ can be replaced by $d\varepsilon_{np} = {}^1/_2 dg\hat{}_{np}$, assuming that g°_{np} is time independent, we can write the first formula of (A.4), as

$$d\Gamma\hat{}^{\alpha}_{mp} = \Gamma\hat{}^{i}_{mp}g\hat{}_{is}dg\hat{}^{\alpha s} + g\hat{}^{\alpha s}(\nabla\hat{}_p d\varepsilon_{ms} + d\varepsilon_{mj}\Gamma\hat{}^{j}_{ps} + d\varepsilon_{sj}\Gamma\hat{}^{j}_{mp} + \nabla\hat{}_m d\varepsilon_{ps} + \tag{A.5}$$

$$+d\varepsilon_{pj}\Gamma^{\wedge j}_{ms}+d\varepsilon_{sj}\Gamma^{\wedge j}_{mp}-\nabla^{\wedge}_{s}d\varepsilon_{mp}-d\varepsilon_{mj}\Gamma^{\wedge j}_{sp}-d\varepsilon_{pj}\Gamma^{\wedge j}_{sm})=$$
$$=\Gamma^{\wedge j}_{mp}g^{\wedge}_{js}dg^{\wedge\alpha s}+g^{\wedge\alpha s}(\nabla^{\wedge}_{p}d\varepsilon_{ms}+\nabla^{\wedge}_{m}d\varepsilon_{ps}-\nabla^{\wedge}_{s}d\varepsilon_{mp})+\Gamma^{\wedge j}_{mp}g^{\wedge\alpha s}dg^{\wedge}_{js}$$

Since

$$\Gamma^{\wedge j}_{mp}g^{\wedge}_{js}dg^{\wedge\alpha s}+\Gamma^{\wedge i}_{mp}g^{\wedge\alpha s}dg^{\wedge}_{js}=\Gamma^{\wedge j}_{mp}d(g^{\wedge\alpha s}g^{\wedge}_{js})=\Gamma^{\wedge j}_{mp}d(\delta_j^{\alpha})=0$$

we finally obtain the formula

$$d\Gamma^{\wedge\alpha}_{mp}=g^{\wedge\alpha s}(\nabla^{\wedge}_{p}d\varepsilon_{ms}+\nabla^{\wedge}_{m}d\varepsilon_{ps}-\nabla^{\wedge}_{s}d\varepsilon_{mp}) \qquad (A.6)$$

Using the latter, we can write (A.2) in the form

$$d\nabla^{\wedge}_{m}\varepsilon_{np}=L^{k\,(i\,j)}_{m\,(np)}\nabla^{\wedge}_{k}d\varepsilon_{ij} \qquad (A.7)$$

It is interesting to note that in the formula obtained previously

$$\nabla^{\wedge}_{m}\varepsilon_{np}=L^{k\,(i\,j)}_{m\,(np)}\nabla^{\circ}_{k}\varepsilon_{ij} \qquad (A.8)$$

and in the formula (A.7), we encounter the same tensor

$$L^{k\,(i\,j)}_{m\,(np)}=\delta_m{}^k\delta_{(n}{}^{(i}\delta_{p)}{}^{j)}-\varepsilon_{n\alpha}g^{\wedge\alpha s}[\delta_p{}^k\delta_m{}^{(i}\delta_s{}^{j)}+\delta_m{}^k\delta_p{}^{(i}\delta_s{}^{j)}-\delta_s{}^k\delta_m{}^{(i}\delta_p{}^{j)}]-$$
$$-\varepsilon_{\alpha p}g^{\wedge\alpha s}[\delta_n{}^k\delta_m{}^{(i}\delta_s{}^{j)}+\delta_m{}^k\delta_n{}^{(i}\delta_s{}^{j)}-\delta_s{}^k\delta_m{}^{(i}\delta_n{}^{j)}] \qquad (A.9)$$

In an analogous manner we obtain the formula

$$d\nabla^{\wedge}_{m}n^{\wedge p}=\nabla^{\wedge}_{m}dn^{\wedge p}+n^{\wedge\alpha}d\Gamma^{\wedge p}_{\alpha m}=\nabla^{\wedge}_{m}dn^{\wedge p}+\Psi^{k\,(ij)\,p}_{m}\nabla^{\wedge}_{k}d\varepsilon_{ij} \qquad (A.10)$$

where

$$\psi^{k\,(ij)\,p}_{m}=\tfrac{1}{2}\delta_m{}^k(n^{\wedge i}g^{\wedge pj}+n^{\wedge j}g^{\wedge pi})+\tfrac{1}{2}n^{\wedge k}(g^{\wedge pj}\delta_m{}^i+g^{\wedge pi}\delta_m{}^j)- \qquad (A.11)$$
$$-\tfrac{1}{2}g^{\wedge pk}(n^{\wedge i}\delta_m{}^j+n^{\wedge j}\delta_m{}^i)$$

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